Space-time covariance functions with compact support

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Abstract

We characterize completely the Gneiting class [6] of space-time covariance functions and give more relaxed conditions on the involved functions. We then show necessary conditions for the construction of compactly supported functions of the Gneiting type. These conditions are very general since they do not depend on the Euclidean norm. Finally, we discuss a general class of positive definite functions, used for multivariate Gaussian random fields. For this class, we show necessary criteria for its generator to be compactly supported.

Keywords : Compact support, Gneiting's class, Positive definite, Space-time.

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1 Introduction

Recent literature persistently emphasizes the use of approximation methods and new methodologies for dealing with massive spatial data set. When dealing with spatial data, calculation of the inverse of the covariance matrix becomes a crucial problem. For instance, the inverse is needed for best linear unbiased prediction (alias kriging), and is repeatedly calculated in the maximum likelihood estimation or the Bayesian inferences. Thus, large spatial sample sizes traduce into big challenges from the computational point of view.

A natural idea that made proselytes in the last year is to make the covariances exactly zero after certain distance so that the resulting matrix has a high proportion of zero entries and is therefore a sparse matrix. Operations on sparse matrices take up less computer memories and run faster. However, this should be done in a way to preserve positive definiteness of the resulting covariance matrix. The idea goes under the name of covariance *tapering*, by meaning that the true covariance is multiplied pointwise with a compactly supported and radial correlation function. This operation is technically justified by the fact that the Schur product preserves positive definiteness.

The effects of tapering in terms of estimation and interpolation have been recently inspected by [4], where general conditions are given in order to ensure that tapering does not affect the efficiency of the maximum likelihood estimator. For spatial interpolation, [5] show that under some regularity conditions, tapering procedures yield asymptotically optimal prediction. In order to assess these properties, the asymptotic framework adopted by the authors is of the infill type, and the tool allowing to evaluate the performances of tapering is the equivalence of Gaussian measures, for which a comprehensive theory can be found in the seminal work by Yadrenko [13].

These points fix very briefly the state of the art and we refer the reader to [3] for an excellent survey on the topic.

Although tapering has been well understood in the spatial framework, there is nothing done, to the knowledge of the authors, for the spatio-temporal case. In particular, the use of tapering is at least questionable in space-time, since the same type of asymptotics does not apply and thus it is not easy to evaluate its performances.

But a deeper look at this problem also highlights the non existence, in the literature, of space-time covariance functions that are compactly supported over space, time or both. These facts motivate the research documented

in this manuscript.

We deal with challenges related to space-time covariance functions. If spatial data set can be massive, one can imagine how the dimensionality problem affects space-time estimation and interpolation. This problem may be faced on the base of two perspectives that can be illustrated through the celebrated T. Gneiting class of covariance functions [6]: for $(x,t) \in \mathbb{R}^{d+l}$, the function

$$(x,y) \mapsto K(x,t) := h(\|t\|^2)^{-d/2} \varphi\left(\frac{\|x\|^2}{h(\|t\|^2)}\right) \tag{1}$$

is positive definite, for φ completely monotone on the positive real line and h a Bernstein function. For $l = 1$, the function above is a stationary and nonseparable space-time covariance. This function has been persistently used by the literature and a Google scholar search highlights that currently there are over 90 papers where this covariance has been used for applications to space-time data.

If there are many observations over space, time or both, then the use of this function would be questionable for the computational reasons exposed above. A more intriguing perspective is to consider a function of the Gneiting type, but replacing the generator φ in equation (1) with a compactly supported function, and inspecting the conditions ensuring that permissibility is preserved on some d-dimensional Euclidean space. The results are illustrated in the following sections.

An auxiliary result of independent interest is also given: we characterize completely the Gneiting class and give more general conditions for its permissibility.

The ratio mentis of this paper leads then to consider a general class of covariances, originally proposed in Porcu *et al.* [10] and more recently in [1]. Both groups of authors show that is class of covariances can be very versatile since it can be used for two-fold purposes: on the one hand, it can be effectively used to deal with zonally anisotropic structures, on the other hand it can be adapted to represent the covariance mapping associated to a multivariate random field, which is highly in demand since there are very few models with these characteristics [7].

As a conclusion to the preludium, the plan of the paper is the following: in Section 2 we present basic facts about positive and negative definite functions. Section 3 characterizes completely the Gneiting class, for which only sufficient conditions were known until now. In Section 4 we present necessary conditions for compactly supported covariances of the Gneiting type. Similar results are obtained in Section 5 for the multivariate class of cross-covariances proposed in [10].

2 Preliminaries

This section is largely expository and contains basic facts and information needed for a self-contained exposition. We shall enunciate the concepts of positive and negative definiteness, as well as the material related to them,

working with linear spaces and subspaces. The space-time notation will be used only when necessary for a clearer exposition of results.

For E a real linear space, we denote by $\mathrm{FD}(E)$ the set of all linear finite-dimensional subspaces of E. If $\dim E = n \in \mathbb{N}$ and $e_1, \ldots, e_n$ are basis in E, then

$$f \in C(E) \quad \Longleftrightarrow \quad f(x_1 e_1 + \ldots + x_n e_n) \in C(R^n)$$

and

$$f \in L(E) \quad \Longleftrightarrow \quad f(x_1 e_1 + \ldots + x_n e_n) \in L(R^n).$$

Also, we call $C_0(E)$ the set of all function $f \in C(E)$ such that f has compact support. If $\dim E = \infty$, then $f \in C(E) \iff f \in C(E_0)\ \forall E_0 \in \mathrm{FD}(E)$.

A complex-valued function $f:\ E \to \mathbb{C}$ is said to be positive definite on E (denoted hereafter $f \in \Phi(E)$) if for any finite collection of points $\{\xi_i\}_{i=1}^n \in E$ the matrix $(f(\xi_i - \xi_j))_{i,j=1}^n$ is positive definite, *i.e.*

$$\text{for all } \ a_1, a_2, \ldots, a_n \in \mathbb{C}\ : \qquad \sum_{i,j=1}^n a_i f(\xi_i - \xi_j)\overline{a}_j \geq 0.$$

It is well known that the family of positive definite functions is a convex cone which is closed under addition, products, pointwise convergence and scale mixtures. Briefly, we have the following properties.

Let $f,\ f_i \in \Phi(E)$, $i \in \mathbb{N}$. Then:

1. $|f(x)| \leq f(0)$, $\overline{f(-x)} = f(x)$, $|f(x) - f(h)|^2 \leq 2f(0)\mathrm{Re}(f(0) - f(x-h))$, $x,\ h \in E$;
2. $\lambda_1 f_1 + \lambda_2 f_2$ with $\lambda_i \geq 0$, $\bar{f}$, $\mathrm{Re}\, f$, $f_1 f_2 \in \Phi(E)$;
3. if, for all $x \in E$, the finite limit $\lim\limits_{n\to\infty} f_n(x) =: g(x)$ exists, then $g \in \Phi(E)$;
4. for any linear operator $A:\ E_1 \to E$ the function $f \circ A$ belongs to $\Phi(E_1)$; in particular, $f \in \Phi(E_1)$ for any linear subspace E_1 from E.

Let $E = \mathbb{R}^n$. The celebrated Bochner's theorem establishes a one to one correspondence between continuous positive definite functions and the Fourier transform of a positive and bounded measure, *i.e.* $f(x) = F_n(\mu(u))(x)$. If μ is absolutely continuous with respect to the Lebesgue measure, than $d\mu(u) = \widehat{f}(u)du$, for $\widehat{f}$ nonnegative. This can be rephrased in the following way: if $f \in C(\mathbb{R}^n) \cap L(\mathbb{R}^n)$, then $f \in \Phi(\mathbb{R}^n)$ if and only if

$$\widehat{f}(u) = F_n^{-1}(f)(u) := \int_{\mathbb{R}^n} e^{i(u,x)} f(x)\ dx \geq 0, \quad u \in \mathbb{R}^n,$$

for $(\cdot,\cdot)$ the usual dot product. The function $\widehat{f}$ is called spectral density or Fourier pair associated to f.

If f is a radially symmetric and continuous function depending on the squared Euclidean norm $\|\cdot\|_2^2$, i.e. $f(x) = \varphi(\|x\|_2^2)$, $\varphi \in C_{[0,+\infty)}$, then the Fourier transform above simplifies to the Bessel integral (if in addition $f \in L(\mathbb{R}^n)$)

$$g_n(s) := \int_0^{+\infty} \varphi(u^2) u^{n-1} j_{\frac{n}{2}-1}(su)\, du\,, \tag{2}$$

where $j_\lambda(u) := \frac{J_\lambda(u)}{u^\lambda}$, with J_λ a Bessel function of the first kind. Thus $f \in \Phi(\mathbb{R}^n)$, for some $n \in \mathbb{N}$ and for f radially symmetric, if and only if $g_n(u) \geq 0 \;\forall u > 0$.

A function $f :]0, \infty[\to \mathbb{R}$ is called *completely monotone*, if it is arbitrarily often differentiable and

$$(-1)^n f^{(n)}(x) \geq 0 \;\text{ for }\; x > 0,\; n = 0, 1, \ldots.$$

By Bernstein's theorem the set $M_{(0,\infty)}$ of completely monotone functions coincides with that of Laplace transforms of positive measures μ on $[0, \infty[$, *i.e.*

$$f(x) = \mathcal{L}\mu(x) = \int_{[0,\infty[} \mathrm{e}^{-xt}\, d\mu(t),$$

where we only require that e^{-xt} is μ-integrable for any $x > 0$. $M_{(0,\infty)}$ is a convex cone which is closed under addition, multiplication and pointwise convergence.

The connection with the function $g_n(\cdot)$ gives the celebrated Schoenberg (1939) theorem by which a radial function $f(x) = \varphi(\|x\|_2^2)$, $\varphi \in C_{[0,+\infty)}$, belongs to $\Phi(\mathbb{R}^n)$ for all $n \in \mathbb{N}$ if and only if φ is completely monotone on the positive real line, and in this case the Bessel integral in equation (2) reduces to a Gaussian mixture. Finally, a Bernstein function is a positive function that is infinitely often differentiable and whose first derivative is completely monotone. For a more detailed exposition on these facts the reader is referred to [11].

In this paper we shall be also dealing with functions depending not on the Euclidean norm but on some homogeneous continuous function $\rho : E \to \mathbb{R}$ such that $\rho(tx) = |t|\rho(x)$ $\forall t \in \mathbb{R}, x \in E$ and $\rho(x) > 0$, $x \neq 0$. If $\varphi \in C_{[0,+\infty)}$ and $\int_0^{+\infty} |\varphi(t^2)| t^{n-1}\, dt < +\infty$, then we have that $\varphi \circ \rho^2 \in \Phi(\mathbb{R}^n)$ if and only if the function

$$\mathbb{R}^n \ni v \mapsto G_n(v) := \int_{\mathbb{R}^n} \varphi(\rho^2(y)) e^{i(y,v)}\, dy \tag{3}$$

is nonnegative for all $v \in \mathbb{R}^n$. If ρ is the Euclidean norm, then the functions $G_n(\cdot)$ and $g_n(\cdot)$ are related by the well know equality $G_n(v) = (2\pi)^{\frac{n}{2}} g_n(||v||_2)$.

Finally, a complex-valued function $h :\; E \to \mathbb{C}$ is called (conditionally) negative definite on E (denoted $h \in N(E)$ hereafter) if the inequality

$$\sum_{k,j=1}^{n} c_k \bar{c}_j h(x_k - x_j) \leq 0$$

is satisfied for any finite systems of complex numbers $c_1, c_2, ..., c_n$, $\sum_{k=1}^{n} c_k = 0$, and points $x_1, ..., x_n$ in E.

Let $\{Z(\xi), \xi \in \mathbb{R}^n\}$ be a continuous weakly stationary and Gaussian random field (RF for short). The associated covariance function $f : \mathbb{R}^n \to \mathbb{R}$ is positive definite. This can be rephrased by saying that positive definiteness of a candidate continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is sufficient condition for the existence of a continuous weakly stationary and Gaussian RF having $f(\cdot)$ as covariance function.

If, additionally, $f(\cdot)$ is *radially symmetric*, the associated Gaussian RF is called *isotropic*. Isotropy and stationarity are independent assumptions but throughout the paper we shall assume both in order to keep things simple.

To complete the picture, the variance of the increments of an intrinsically stationary Gaussian RF is called variogram. For two points of $\mathbb{R}^n$, say ξ_i, $i = 1, 2$, we have that $\mathbb{V}\mathrm{ar}\,(Z(\xi_2) - Z(\xi_1)) := \gamma(\xi_2 - \xi_1)$. The mapping $\gamma(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is conditionally negative definite. The additional property of isotropy is then analogously defined as before.

3 Complete Characterization of the Gneiting class

Lemma 1.

i. $f \in \Phi(E) \iff f \in \Phi(E_0)\ \forall E_0 \in \mathrm{FD}(E)$.

ii. *If* $\dim E = n \in \mathbb{N}$ *then* $f \in \Phi(E) \iff fg \in \Phi(E)\ \forall g \in \Phi(E) \cap C_0(E)$.

Proof.

i. The necessity is obvious. As for the sufficiency, for $n \in \mathbb{N}$ and $x_1, ..., x_n$ in E, we have that $x_1, ..., x_n \in E_0$ - the linear span of these elements. Obviously $\dim E_0 \leq n$.

ii. Again, the necessity is obvious. For the sufficiency, let $e_1, \dots, e_n$ be basis in E. Then we take $g(x_1 e_1 + \dots + x_n e_n) = (1 - \varepsilon|x_1|)_+ \cdot \dots \cdot (1 - \varepsilon|x_n|)_+$ and $\varepsilon \downarrow 0$. The proof is completed.

□

Lemma 2. *Let the next conditions be satisfied:*

1. $h, b \in C(E)$ *and* $h(t) > 0\ \forall t \in E$.

2. $\varphi \in C_{[0,+\infty)}$ *and for the some* $m \in \mathbb{N}$: $\int_0^{+\infty} |\varphi(u^2)| u^{m-1}\, du < +\infty$.

3. $\rho \in C(\mathbb{R}^m)$, $\rho(tx) = |t|\rho(x)\ \forall t \in \mathbb{R}, x \in \mathbb{R}^m$ *and* $\rho(x) > 0$, $x \neq 0$.

Then

$$K(x,t) := b(t)\varphi\left(\frac{\rho^2(x)}{h(t)}\right) \in \Phi(\mathbb{R}^m \times E) \iff b(t)(h(t))^{\frac{m}{2}} G_m(\sqrt{h(t)}v) \in \Phi(E)\ \forall v \in \mathbb{R}^m\,,$$

with $G_m(\cdot)$ defined in equation (3).

Proof. Observe that $\varphi\left(\rho^2(x)\right) \in L(\mathbb{R}^m)$. We have that

$$\begin{aligned}
K(x,t) \in \Phi(\mathbb{R}^m \times E) \quad &\Longleftrightarrow \quad K(x,t) \in \Phi(\mathbb{R}^m \times E_0) \ \ \forall E_0 \in \mathrm{FD}(E) \\
&\Longleftrightarrow \quad K(x,t)g(t) \in \Phi(\mathbb{R}^m \times E_0) \ \ \forall E_0 \in \mathrm{FD}(E), \forall g \in \Phi(E_0) \cap C_0(E_0) \\
&\Longleftrightarrow \quad \iint_{\mathbb{R}^m \times E_0} K(x,t)g(t)e^{i(x,v)}e^{i(t,u)}dxdt \geq 0 \\
& \qquad\quad \forall E_0 \in \mathrm{FD}(E), \forall g \in \Phi(E_0) \cap C_0(E_0) \ , \ \forall v \in \mathbb{R}^m, u \in E_0.
\end{aligned}$$

As for the last integral, a change of variables of the type $x = \sqrt{h(t)}y$ yields that the last inequality is equivalent to

$$\int_{E_0} g(t)b(t)(h(t))^{\frac{m}{2}} G_m(\sqrt{h(t)}v)e^{i(t,u)}dt \geq 0 \ , \ \forall v \in \mathbb{R}^m, u \in E_0,$$

which holds if, and only if $\forall g \in \Phi(E_0) \cap C_0(E_0), \forall v \in \mathbb{R}^m$, we have

$$\begin{aligned}
& g(t)b(t)(h(t))^{\frac{m}{2}} G_m(\sqrt{h(t)}v) \in \Phi(E_0) \ \ \forall E_0 \in \mathrm{FD}(E), \\
\Longleftrightarrow \quad & b(t)(h(t))^{\frac{m}{2}} G_m(\sqrt{h(t)}v) \in \Phi(E) \ \forall v \in \mathbb{R}^m .
\end{aligned}$$

The proof is completed. □

The following result gives a complete characterization of the Gneiting class, with the additional feature that only negative definiteness of the function h is required, whilst Gneiting's assumptions are much more restrictive as it is required that h' is completely monotone on the positive real line. Furthermore, we give a simple proof of this result and we defer it to next section for the reasons that will become apparent throughout the paper.

Theorem 3. *Let $h \in C(E)$, $h(t) > 0$ $\forall t \in E$. Let $d \in \mathbb{N}$. The following statements are equivalent:*

1. $K(x,t) := (h(t))^{-\frac{d}{2}} \varphi\left(\frac{||x||_2^2}{h(t)}\right) \in \Phi(\mathbb{R}^d \times E) \ \ \forall \varphi \in C_{[0,+\infty)} \bigcap M_{(0,+\infty)}$.
2. $e^{-\lambda h(t)} \in \Phi(E)$ $\forall \lambda > 0$.

Let us consider examples of functions h for which the statement 2 in Theorem 3 holds.

Example 1 Let $h(t) = ||t||_p^\alpha + c$, $c > 0$, $0 < p \leq +\infty$, $\alpha \geq 0$, $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, where $||t||_p^p = \sum_{k=1}^n |t_k|^p$, $0 < p < \infty$, and $||t||_\infty = \sup_{1 \leq k \leq n} |t_k|$. Then

$$e^{-\lambda h(t)} \in \Phi(\mathbb{R}^n) \ \forall \lambda > 0 \iff e^{-||t||_p^\alpha} \in \Phi(\mathbb{R}^n) \iff 0 \leq \alpha \leq \alpha(l_p^n) \ ,$$

where

$$\alpha(l_p^n) = \begin{cases} 2 & \text{if } n = 1, \ 0 < p \leq \infty; \\ p & \text{if } n \geq 2, \ 0 < p \leq 2; \\ 1 & \text{if } n = 2, \ 2 < p \leq \infty; \\ 0 & \text{if } n \geq 3, \ 2 < p \leq \infty. \end{cases} \tag{4}$$

For $0 < p \leq 2$, we get Schoenberg's result. The other two cases have been investigated by Koldobsky [8] in 1991 and Zastavnyi [14, 15, 16] in 1991 ($2 < p \leq \infty$, $n \geq 2$). Finally, Misiewiez [9] gave the last result in 1989 ($p = \infty$, $n \geq 3$).

Example 2 If $\rho(t)$ is a norm on $\mathbb{R}^2$, then $e^{-\rho^\alpha(t)} \in \Phi(\mathbb{R}^2)$ for all $0 \leq \alpha \leq 1$. This is a well-know fact (see, for example, [17]). Therefore $e^{-\lambda h(t)} \in \Phi(\mathbb{R}^2)$ $\forall \lambda > 0$, where $h(t) = \rho^\alpha(t) + c$, $0 \leq \alpha \leq 1$, $c > 0$.

Example 3 Let $\psi(s) \in \mathbb{R}$ $\forall s > 0$. Then, it is well known that

$$e^{-\lambda\psi} \in M_{(0,+\infty)} \; \forall \lambda > 0 \iff \psi' \in M_{(0,+\infty)}.$$

Gneiting [6] proves the following: if $\psi \in C_{[0,+\infty)}$, $\psi(s) > 0$ $\forall s \geq 0$, and $\psi' \in M_{(0,+\infty)}$, then $e^{-\lambda h(t)} \in \Phi(\mathbb{R}^n)$ for all $\lambda > 0$, $n \in \mathbb{N}$, where $h(t) := \psi(||t||_2^2)$ and, hence,

$$K(x,t) := (\psi(||t||_2^2))^{-\frac{d}{2}} \varphi\left(\frac{||x||_2^2}{\psi(||t||_2^2)}\right) \in \Phi(\mathbb{R}^d \times \mathbb{R}^n) \; \forall \varphi \in C_{[0,+\infty)} \bigcap M_{(0,+\infty)}\,,\; d \in \mathbb{N}\,.$$

Example 4 By celebrated Schoenberg's Theorem [12], if $h(-t) = h(t)$ $\forall t \in E$, then

$$h(t) \in N(E) \iff e^{-\lambda h(t)} \in \Phi(E) \; \forall \lambda > 0\,.$$

Example 5 Let $g \in \Phi(E)$, $g(-t) = g(t)$ for all $t \in E$ and $h(t) := g(0) - g(t) + c$, $c > 0$. Then $h(t) > 0$ $\forall t \in E$, $h \in N(E)$ and, hence, $e^{-\lambda h(t)} \in \Phi(E)$ for all $\lambda > 0$.

4 Necessary conditions for compactly supported functions of the Gneiting type

From now on let us write $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : ||x||_2 = 1\}$ for the sphere of $\mathbb{R}^d$.

Theorem 4. *Let the next conditions be satisfied:*

1) $h \in C(E)$, $h(t) > 0$ $\forall t \in E$ *and* $h(t) \not\equiv h(0)$ *on* E.

2) $\varphi \in C_{[0,+\infty)}$, $\varphi(0) > 0$.

3) For $d \in \mathbb{N}$, $\rho \in C(\mathbb{R}^d)$, $\rho(tx) = |t|\rho(x)$ $\forall t \in \mathbb{R}, x \in \mathbb{R}^d$ and $\rho(x) > 0$, $x \neq 0$.

4) $K(x,t) := (h(t))^{-\frac{d}{2}}\varphi\left(\frac{\rho^2(x)}{h(t)}\right) \in \Phi(\mathbb{R}^d \times E)$.

Then:

1. *$(h(t))^{-\frac{d}{2}} \in \Phi(E)$ and $\varphi\left(\rho^2(x)\right) \in \Phi(\mathbb{R}^d)$.*

2. *If there exists a $n \in \mathbb{N}\bigcap[1,d]$ such that $\int_0^{+\infty} |\varphi(u^2)|u^{n-1}\,du < +\infty$, then $\forall m = 1,\dots,n$ and $v \in \mathbb{R}^m$ the function $s \mapsto f_{m,v}(s) := s^{m-d}G_m(sv)$, with $G_m(\cdot)$ as defined in (3), is decreasing on $(0,+\infty)$. Furthermore, $f_{m,v}(+\infty) = 0$ for $v \neq 0$.*

3. *If $\int_0^{+\infty} |\varphi(u^2)|u^{d-1}\,du < +\infty$, then $G_d(0) > 0$. If, in addition, G_d is real-analytic, then $\forall v \in \mathbb{R}^d$, $v \neq 0$ the function $s \mapsto f_{d,v}(s) := G_d(sv)$ is strictly decreasing on $[0,+\infty)$ and $G_d(v) > 0$ $\forall v \in \mathbb{R}^d$.*

4. *If $\int_0^{+\infty} |\varphi(u^2)|u^{d+1}\,du < +\infty$, then $\alpha_1(v) := \int_{\mathbb{R}^d} \varphi(\rho^2(y))(y,v)^2\,dy \geq 0$ $\forall v \in \mathbb{S}^{d-1}$ and $\beta_1 := \int_{\mathbb{R}^d} \varphi(\rho^2(y))||y||_2^2\,dy \geq 0$. Furthermore, $\alpha_1(v) \equiv 0$ on $\mathbb{S}^{d-1} \iff \beta_1 = 0$. If, in addition, $\beta_1 > 0$, then $e^{-\lambda h(t)} \in \Phi(E)$ $\forall \lambda > 0$.*

5. *If $\int_0^{+\infty} |\varphi(u^2)|e^{\varepsilon u}\,du < +\infty$ for some $\varepsilon > 0$ (for example, when φ has compact support), then $\exists p \in \mathbb{N}$: $e^{-\lambda h^p(t)} \in \Phi(E)$ $\forall \lambda > 0$. In practice, for p it is possible to take one of the following numbers:*

$$p(v) := \min\left\{k \in \mathbb{N} : \ \alpha_k(v) = \int_{\mathbb{R}^d} \varphi(\rho^2(y))(y,v)^{2k}\,dy \neq 0\right\} \ , \ v \in \mathbb{S}^{d-1} \ ,$$

$$q := \min\left\{k \in \mathbb{N} : \ \beta_k = \int_{\mathbb{R}^d} \varphi(\rho^2(y))||y||_2^{2k}\,dy \neq 0\right\} \ .$$

The function $p(\cdot)$ is bounded on $\mathbb{S}^{d-1}$ and $q = \min\limits_{v\in\mathbb{S}^{d-1}} p(v)$.

Proof. The statement **1** is obvious.

Let us prove the statement **2.** By Lemma 2, we have

$$F_{m,v}(t) := (h(t))^{\frac{m-d}{2}}G_m(\sqrt{h(t)}v) \in \Phi(E) \ , \ \forall m = \overline{1,n} \ , \ v \in \mathbb{R}^m \ .$$

Hence, $F_{m,v}(0) = (h(0))^{\frac{m-d}{2}}G_m(\sqrt{h(0)}v) \geq 0$ and $|F_{m,v}(t)| \leq F_{m,v}(0)$, $t \in E$. Therefore $G_m(v) \geq 0$, $v \in \mathbb{R}^m$, and

$$(sh(t))^{\frac{m-d}{2}}G_m(\sqrt{h(t)}sv) \leq (sh(0))^{\frac{m-d}{2}}G_m(\sqrt{h(0)}sv) \ , \ \forall m = \overline{1,n} \ , \ v \in \mathbb{R}^m \ , \ s > 0 \ , \ t \in E \, .$$

The latter inequality is equivalent to

$$f_{m,v}\left(\sqrt{\frac{h(t)}{h(0)}} \cdot s\right) \leq f_{m,v}\left(s\right) \ , \ \forall m = \overline{1,n} \ , \ v \in \mathbb{R}^m \ , \ s > 0 \ , \ t \in E \, .$$

Since $(h(t))^{-\frac{d}{2}} \in \Phi(E)$, then $h(t) \geq h(0)$, $t \in E$. Since $h(t) \not\equiv h(0)$ on E, then there exists a point $t_0 \in E$ such that $q := \sqrt{\frac{h(t_0)}{h(0)}} > 1$. By the intermediate values Theorem $\forall \alpha \in [1,q]$ $\exists \xi \in E : \sqrt{\frac{h(\xi)}{h(0)}} = \alpha$. Therefore,

$f_{m,v}(\alpha s) \le f_{m,v}(s)$ for all $s > 0$ and $\alpha \in [1, q]$. Hence, $f_{m,v}(\alpha^2 s) \le f_{m,v}(\alpha s) \le f_{m,v}(s)$ for all $s > 0$ and $\alpha \in [1, q]$. Thus, $f_{m,v}(\alpha^p s) \le f_{m,v}(s)$ for all $s > 0$, $\alpha \in [1, q]$ and $p \in \mathbb{N}$. This implies that the function $f_{m,v}(s)$ decreases in $s \in (0, +\infty)$. By the Riemann-Lebesgue Theorem, it follows that $G_m(v) \to 0$ as $||v||_2 \to +\infty$. Hence $f_{m,v}(+\infty) = 0$ for $v \neq 0$. The statement **2** is proved.

Let us prove the statement **3.** **i**. From statement **2** it follows that for all $v \in \mathbb{R}^d$, $v \neq 0$, the function $G_d(sv)$ decreases in $s \in [0, +\infty)$ and, hence, $0 \le G_d(v) \le G_d(0)$. Therefore, $G_d(0) > 0$ (otherwise $G_d(v) \equiv 0$ on $\mathbb{R}^d \Rightarrow \varphi(\rho^2(y)) \equiv 0$ on $\mathbb{R}^d$, that contradicts the condition $\varphi(0) > 0$).

ii. If, in addition, G_d is real-analytic, then $\forall v \in \mathbb{R}^d$, $v \neq 0$, the function $G_d(sv)$ strictly decreases on $[0, +\infty)$. This can be proved by contraddiction. Let us assume that, for some $v_0 \in \mathbb{R}^d$ and $v_0 \neq 0$, the function $G_d(sv_0)$ is constant on some interval $(\alpha, \beta) \subset (0, +\infty)$, $\alpha < \beta$. This would imply that G_d it is constant on $[0, +\infty)$ and $G_d(0) = \lim_{s\to+\infty} G_d(sv_0) = 0$, which contradicts **i**. Thus, $\forall v \in \mathbb{R}^d$, $v \neq 0$, the function $G_d(sv)$ strictly decreases on $[0, +\infty)$ and, hence, $G_d(v) > \lim_{s\to+\infty} G_d(sv) = 0$. The statement **3** is proved.

Let us prove statement **4.** Let $v \in \mathbb{S}^{d-1}$ and $f_{d,v}(s) := G_d(sv)$. From statements **2** and **3**, it follows that the function $f_{d,v}(s)$ decreases on $[0, +\infty)$ and that $f_{d,v}(0) > 0$. Obviously, $f_{d,v}(s) \in C^2(\mathbb{R})$ and

$$f_{d,v}(s) = f_{d,v}(0) + \frac{f''_{d,v}(0)}{2}\, s^2 + o(s^2)\ ,\ s \to 0\,,$$

where $f''_{d,v}(0) = -\alpha_1(v)$. Note that $f''_{d,v}(0) \le 0$, otherwise the function $f_{d,v}(s)$ strongly increases on $[0, c]$ for some $c > 0$, which contradicts statement **2**. Thus, $\alpha_1(v) \ge 0$ for all $v \in \mathbb{S}^{d-1}$. For $p > 0$, the next integral is constant on $\mathbb{S}^{d-1}$:

$$\int_{\mathbb{S}^{d-1}} |(y, v)|^p \, d\sigma(v) \equiv c_{d,p} > 0\ ,\ y \in \mathbb{S}^{d-1}\ ,$$

where $d\sigma$, if $n \ge 2$, is the surface measure on $\mathbb{S}^{d-1}$ and $d\sigma(v) = \delta(v-1) + \delta(v+1)$, if $d = 1$ (here $\delta(v)$ - the Dirac measure with mass 1 concentrated in the point $v = 0$). Therefore,

$$\int_{\mathbb{S}^{d-1}} |(y, v)|^p \, d\sigma(v) = c_{d,p} ||y||_2^p\ ,\ y \in \mathbb{R}^d\ , p > 0. \tag{5}$$

Hence

$$\int_{\mathbb{S}^{d-1}} \alpha_1(v)\, d\sigma(v) = c_{d,2}\, \beta_1 \ge 0$$

and $\alpha_1(v) \equiv 0$ on $\mathbb{S}^{d-1} \iff \beta_1 = 0$.

Let, in addition, $\beta_1 > 0$. Then $f''_{d,v_0}(0) = -\alpha_1(v_0) < 0$ for some $v_0 \in \mathbb{S}^{d-1}$ and

$$\psi_n(t) := \left(\frac{G_d(\gamma_n \sqrt{h(t)}\, v_0)}{G_d(0)} \right)^n = (1 + g_n(t))^n \in \Phi(E)\ ,\ \forall n \in \mathbb{N}\,, \gamma_n > 0. \tag{6}$$

Take

$$\gamma_n := \left(-\frac{2 f_{d,v_0}(0)}{f''_{d,v_0}(0)} \cdot \frac{\lambda}{n} \right)^{\frac{1}{2}} > 0\ ,\ \lambda > 0.$$

Obviously, $\gamma_n \to +0$ and

$$g_n(t) = \frac{f_{d,v_0}(\gamma_n\sqrt{h(t)}) - f_{d,v_0}(0)}{f_{d,v_0}(0)} \sim \frac{f''_{d,v_0}(0)}{2f_{d,v_0}(0)} \cdot (\gamma_n\sqrt{h(t)})^2 = -\frac{\lambda}{n} \cdot h(t)\ ,\ n \to \infty\,.$$

Therefore, $\psi_n(t) \to e^{-\lambda h(t)}$ and, hence, $e^{-\lambda h(t)} \in \Phi(E)$ for all $\lambda > 0$. The statement **4** is proved.

Let us prove the statement **5**. In this case G_d is real-analytic and

$$f^{(2k)}_{d,v}(0) = (-1)^k \alpha_k(v)\ ,\ f^{(2k-1)}_{d,v}(0) = 0\ ,\ \int_{\mathbb{S}^{d-1}} \alpha_k(v)\, d\sigma(v) = c_{d,2k}\, \beta_k\ ,\ k \in \mathbb{N}\,. \tag{7}$$

Therefore, $\forall v \in \mathbb{S}^{d-1}\ \exists p \in \mathbb{N}$ so that

$$f_{d,v}(s) = f_{d,v}(0) + \frac{f^{(2p)}_{d,v}(0)}{(2p)!}\, s^{2p} + o(s^{2p})\ ,\ s \to 0\,,$$

where $f^{(2p)}_{d,v}(0) \neq 0$, otherwise the function $f_{d,v}(0) \equiv f_{d,v}(s) \equiv f_{d,v}(+\infty) = 0$ which contradicts the inequality $G_d(0) > 0$ (see statement **3**). Hence, $f^{(2p)}_{d,v}(0) < 0$, otherwise the function $f_{d,v}(s)$ strongly increases on $[0, c]$ for some $c > 0$, which contradicts statement **2**. Thus the function $p(v)$, $v \in \mathbb{S}^{d-1}$, defines correctly.

Let $v \in \mathbb{S}^{d-1}$ and $p = p(v)$. Take function (6), where $v_0 = v$

$$\gamma_n := \left(-\frac{(2p)!f_{d,v_0}(0)}{f^{(2p)}_{d,v_0}(0)} \cdot \frac{\lambda}{n}\right)^{\frac{1}{2p}} > 0\ ,\ \lambda > 0.$$

Then $g_n(t) \sim -\frac{\lambda}{n} \cdot h^p(t)$, $n \to \infty$. Therefore $\psi_n(t) \to e^{-\lambda h^p(t)}$ and, hence, $e^{-\lambda h^p(t)} \in \Phi(E)$ for all $\lambda > 0$.

If $\alpha_k(v_0) \neq 0$ for some $v_0 \in \mathbb{S}^{d-2}$, $k \in \mathbb{N}$, then $\alpha_k(v) \neq 0$ in some neighborhood of a point v_0 and, hence, $p(v) \leq p(v_0)$ in this neighborhood. Thus the function $p(v)$ is locally bounded on compact $\mathbb{S}^{d-1}$ and, hence, $p(v)$ is bounded on $\mathbb{S}^{d-1}$.

Let $m = \min\limits_{v \in \mathbb{S}^{d-1}} p(v) = p(v_0)$ for some $v_0 \in \mathbb{S}^{d-1}$. Then $\alpha_m(v_0) \neq 0$ and for all $v \in \mathbb{S}^{d-1}$ equality

$$f_{d,v}(s) = f_{d,v}(0) + \frac{f^{(2m)}_{d,v}(0)}{(2m)!}\, s^{2m} + o(s^{2m})\ ,\ s \to 0$$

holds. Obviously $(-1)^k \alpha_k(v) = f^{(2k)}_{d,v}(0) = 0$, for all $1 \leq k < m$ (if $m \geq 2$), and $(-1)^m \alpha_m(v) = f^{(2m)}_{d,v}(0) \leq 0$ (otherwise the function $f_{d,v}(s)$ strongly increases on $[0, c]$ for some $c > 0$ that contradicts a statement **2**). From (7) follows that $\beta_k = 0$ for all $1 \leq k < m$ (if $m \geq 2$) and $(-1)^m \beta_m < 0$. Therefore $q = m$.

The Theorem 4 is proved. $\square$

Proof of Theorem 3. If $h(t) \equiv h(0) > 0$ on E, then the implication 1) $\Rightarrow$ 2) is obvious. If $h(t) \not\equiv h(0)$ on E, then this implication follows from statement **4** of Theorem 4 for $\varphi(s) = e^{-s} \in C_{[0,+\infty)} \bigcap M_{(0,+\infty)}$.

The reverse implication 2) $\Rightarrow$ 1) follows from Lemma 2 for $\varphi(s) = e^{-s}$, equality

$$\int_{\mathbb{R}^d} e^{-\frac{1}{2\sigma}||y||_2^2}\, e^{i(y,v)}\, dy = (2\pi\sigma)^{\frac{d}{2}}\, e^{-\frac{\sigma}{2}||v||_2^2}\ ,\ v \in \mathbb{R}^d\,,\ \sigma > 0\,,$$

and Bernstein-Widder's Theorem. The proof is completed. $\square$

Next Theorem 5 is an addition to Theorem 4 for the case $\rho(x) = ||x||_2$.

Theorem 5. *Let the next conditions be satisfied:*

1) $h \in C(E)$, $h(t) > 0$ $\forall t \in E$ *and* $h(t) \not\equiv h(0)$ *on* E.

2) $\varphi \in C_{[0,+\infty)}$, $\varphi(0) > 0$.

3) $K(x,t) := (h(t))^{-\frac{d}{2}}\varphi\left(\frac{||x||_2^2}{h(t)}\right) \in \Phi(\mathbb{R}^d \times E)$.

If $\int_0^{+\infty} |\varphi(u^2)|u^{m-1}\,du < +\infty$ *for some natural* $m \in [1,d]$ *and* g_m *is real-analytic, then the function* $f_m(s) := s^{m-d}g_m(s)$ *strictly decreases on* $(0,+\infty)$ *and* $g_m(s) > 0$ *for all* $s > 0$.

Proof. From Theorem 4 it follows that f_m decreases on $(0,+\infty)$ and $f_m(s) \geq f_m(+\infty) = 0$ for $s > 0$. Since f_m is real-analytic on $(0,+\infty)$, then function $f_m(s)$ strictly decreases on $(0,+\infty)$. Otherwise the function f_m is constant on some interval $(\alpha,\beta) \subset (0,+\infty)$, $\alpha < \beta$, and, hence, it is constant on $(0,+\infty)$ and $f_m(s) = f_m(+\infty) = 0$, $s > 0$. Therefore, $G_m(v) = (2\pi)^{\frac{m}{2}}g_m(||v||_2) \equiv 0$ on $\mathbb{R}^m$. Hence, $\varphi(||x||_2^2) \equiv 0$ on $\mathbb{R}^m$, which contradicts the condition $\varphi(0) > 0$. Thus, the function f_m strictly decreases on $(0,+\infty)$ and, hence, $f_m(s) > f_m(+\infty) = 0$ for all $s > 0$. The Theorem 5 is proved. ◻

5 Some statements involving a versatile general covariance function

Previous results can be generalized to the class of positive definite functions built in [10] and used for the purposes highlighted in Section 1.

Lemma 6. *Let* $\varphi \in C\left([0,+\infty)^n\right)$, $n \in \mathbb{N}$, *and* $\int_0^\infty \ldots \int_0^\infty \left|\varphi(u_1^2,\ldots,u_n^2)\right| \prod_{k=1}^n u_k^{d_k-1}\,du_1\ldots du_n < +\infty$ *for some* $d_k \in \mathbb{N}$, $k = 1,\ldots,n$. *Let* $h_k, b_k \in C(E_k)$, h_k *strictly positive in their arguments for all* $k = 1,\ldots,n$. *Then*

$$K(x_1,\ldots,x_n,t_1,\ldots,t_n) := \varphi\left(\frac{\|x_1\|_2^2}{h_1(t_1)},\ldots,\frac{\|x_n\|_2^2}{h_n(t_n)}\right)\prod_{k=1}^n b_k(t_k) \in \Phi(\mathbb{R}^{d_1}\times\ldots\times\mathbb{R}^{d_n}\times E_1\times\ldots\times E_n)$$

if, and only if,

$$g_{d_1,\ldots,d_n}\left(s_1\sqrt{h_1(t_1)},\ldots,s_n\sqrt{h_n(t_n)}\right)\prod_{k=1}^n b_k(t_k)\big(h_k(t_k)\big)^{d_k/2} \in \Phi(E_1\times\ldots\times E_n)$$

for every $s_k \geq 0$, $k = 1,\ldots,n$, *where*

$$g_{d_1,\ldots,d_n}(s_1,\ldots,s_n) := \int_0^\infty \ldots \int_0^\infty \varphi(u_1^2,\ldots,u_n^2)\prod_{k=1}^n u_k^{d_k-1} j_{d_k/2-1}(s_k u_k)\,du_1\ldots du_n.$$

Proof. The statement can be proved in a similar way as Lemma 2. ◻

Let P_+^n be the set all finite nonnegative Borel measures on $[0,+\infty)^n$, $n \in \mathbb{N}$, and

$$\mathcal{L}^n := \left\{\varphi(u_1,\ldots,u_n) = \int_0^\infty \ldots \int_0^\infty e^{-(u_1v_1+\ldots+u_nv_n)}\,d\mu(v_1,\ldots,v_n)\ ,\ \mu \in P_+^n\right\}\ .$$

Obviously, $\mathcal{L}^1 = C_{[0,+\infty)} \bigcap M_{(0,+\infty)}$ and $\prod_{k=1}^n \varphi_k(u_k) \in \mathcal{L}^n$ for every $\varphi_k \in \mathcal{L}^1$, $k = 1,\ldots,n$.

Theorem 7. *Let $n \in \mathbb{N}$. For all $k = 1, \ldots, n$, let E_k be linear spaces, h_k strictly positive functions such that $h_k \in C(E_k)$ and $d_k \in \mathbb{N}$. Then, the following statements are equivalent:*

1. $K(x_1, \ldots, x_n, t_1, \ldots, t_n) := \varphi\left(\frac{\|x_1\|_2^2}{h_1(t_1)}, \ldots, \frac{\|x_n\|_2^2}{h_n(t_n)}\right) \prod_{k=1}^n \left(h_k(t_k)\right)^{-d_k/2} \in \Phi(\mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_n} \times E_1 \times \ldots \times E_n)$ $\forall \varphi \in \mathfrak{L}^n$.

2. $e^{-\lambda h_k(t_k)} \in \Phi(E_k)$, $\forall \lambda > 0$, $k = 1, \ldots, n$.

Proof. Let us prove the implication (1) $\Longrightarrow$ (2). For every fixed $k = 1, \ldots, n$ in condition **1**, we take $\varphi(u_1, \ldots, u_n) = \varphi_k(u_k)$, $\varphi_k \in \mathfrak{L}^1$, and $t_i = 0 \in E_i$ for $i \neq k$. Then $\varphi_k\left(\frac{||x_k||_2^2}{h_k(t_k)}\right) (h_k(t_k))^{-d_k/2} \in \Phi(\mathbb{R}^{d_k} \times E_k)$ $\forall \varphi_k \in \mathfrak{L}^1$. By Theorem 3 we get $e^{-\lambda h_k(t_k)} \in \Phi(E_k)$ $\forall \lambda > 0$.

Let us now prove the reverse implication. Let $e^{-\lambda h_k(t_k)} \in \Phi(E_k)$, $\forall \lambda > 0$, $k = 1, \ldots, n$. By Theorem 3, we have that $\varphi_k\left(\frac{||x_k||_2^2}{h_k(t_k)}\right) (h_k(t_k))^{-d_k/2} \in \Phi(\mathbb{R}^{d_k} \times E_k)$ $\forall \varphi_k \in \mathfrak{L}^1$, $k = 1, \ldots, n$. We take $\varphi_k(u_k) = e^{-u_k v_k}$, $v_k \geq 0$. From definition of class $\mathfrak{L}^n$ follows, that $\varphi\left(\frac{\|x_1\|_2^2}{h_1(t_1)}, \ldots, \frac{\|x_n\|_2^2}{h_n(t_n)}\right) \prod_{k=1}^n \left(h_k(t_k)\right)^{-d_k/2} \in \Phi(\mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_n} \times E_1 \times \ldots \times E_n)$ $\forall \varphi \in \mathfrak{L}^n$. $\square$

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